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AN EASIER SUPERRIGID COUNTABLE T_1-SPACE

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A space X is superrigid if every self-map of X is the identity of X or is constant. In this paper we construct a simple example of a superrigid countable T_1-space.

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rigid superrigid T_1-space

In [1] Barendregt and van Mill construct a countable T_1-space X that is *superrigid*, i.e., every *self-map* of X ($\equiv$continuous function $X \to X$) is the identity of X or is constant. We simplify that construction.

Call two spaces X and Y *orthogonal* if for every open subspace U of X every map $U \to Y$ is constant, and vice versa. The construction in [1] consists of two parts:

Part 1. There is a countably infinite family $\mathcal{F}$ of countably infinite compact anti-Hausdorff spaces such that every two distinct members of $\mathcal{F}$ are orthogonal. (As usual, we call a space X *anti-Hausdorff* if every two nonempty open sets in X intersect.)

Part 2. The spaces of Part 1 are combined to get a superrigid countable T_1-space.

Barendregt and van Mill call Part 1 surprisingly complicated. We find Part 2 surprisingly complicated, and show how it can be simplified. Part 2, as done in [1], is not unlike Herrlich's construction of regular spaces X with the property that every continuous function $X \to Y$ is constant [3]; that is a recursive construction of length ω. We use a modification of the one-step construction in [2] of such a space (for $Y = \mathbb{R}$).

We now proceed to the simplification of Part 2.

Let $\langle A_P: P \in [\omega]^2 \rangle$ be a pairwise disjoint collection of infinite subsets of ω. For $P \in [\omega]^2$ let $X_P = A_P \cup P$. By Part 1 it is possible to topologize the X_P in such a way that

(1) $(\forall P \in [\omega]^2)$ [X_P is a compact anti-Hausdorff T_1-space]; and

(2) $(\forall P \neq Q \in [\omega]^2)$ [X_P and X_Q are orthogonal].

Now let X be ω topologized by the rule

(3) $(\forall U \subseteq \omega)$ [U is open in $X \Leftrightarrow (\forall P \in [\omega]^2)$ [$U \cap X_P$ is open in X_P]], or, equivalently,

(4) $(\forall F \subseteq U)$ [F is closed in $X \Leftrightarrow (\forall P \in [\omega]^2)$ [$F \cap X_P$ is closed in X_P]].

From (4) it is clear that X is T_1.

Before we prove every self-map of X is constant or is the identity we make some observations about X.

(a) $(\forall P \neq Q \in [\omega]^2)$ $[X_P \cap X_Q$ is finite$]$.

(b) $(\forall P \in [\omega]^2)$ $[X_P$ is closed in $X]$.

(c) $(\forall P \in [\omega]^2)$ $[X_P$ is nowhere dense in $X]$.

(a) is clear, (b) follows from (a) since the X_P are T_1, and (c) follows from (a) and (b) since each X_P is a T_1-space without isolated points.

(d) For every compact $K \subseteq X$ there is a finite $\mathscr{P} \subseteq [\omega]^2$ such that $K \subseteq \bigcup\{X_P: P \in \mathscr{P}\}$.

(This is a well-known property of spaces topologized by a rule like (3).) Indeed, if $K \subseteq \bigcup\{X_P: P \in \mathscr{P}\}$ for every finite $\mathscr{P} \subseteq [\omega]^2$, then there is an infinite $I \subseteq K$ such that $(\forall P \in [\omega]^2)$ $[|I \cap X_P| < \omega]$, hence every subset of I is closed in X, because of (4), hence I is closed discrete in X, hence K is not a compact subspace of X.

The crux of the matter is the following:

Claim. *For every $Q \in [\omega]^2$, if f is a map $X_Q \to X$ with* $\operatorname{ran}(f) \subseteq X_Q$, *then f is constant.*

Indeed, consider a $Q \in [\omega]^2$ and a map $f: X_Q \to X$ with $\operatorname{ran}(f) \subseteq X_Q$. Since X_Q is compact, so is $\operatorname{ran}(f)$. Hence by (d) there is a finite $\mathscr{P} \subseteq [\omega]^2$ such that $\operatorname{ran}(f)$ is a subset of the subspace

$$Y = \bigcup\{X_P: P \in \mathscr{P}\}$$

of X. We may assume without loss of generality that $\operatorname{ran}(f)$ is not a subset of the finite set $\bigcup \mathscr{P}$. (For X_Q is connected, being anti-Hausdorff, hence $\operatorname{ran}(f)$ is connected, but every finite connected subset of the T_1-space X is a singleton.) Since $\operatorname{ran}(f) \not\subseteq X_0$ it follows that there is $R \in \mathscr{P}$ with $R \neq Q$ such that $\operatorname{ran}(f) \cap (X_R \setminus \bigcup \mathscr{P}) \neq \emptyset$. We claim $X_R \setminus \bigcup \mathscr{P}$ is open in Y. Indeed, the set

$$A = \mathscr{P} \cup \bigcup\{X_P: P \in \mathscr{P} \setminus \{R\}\}$$

is closed in X, because of (a) and the fact that X is T_1. As $X_R \setminus \bigcup \mathscr{P} = Y \setminus A$ this shows that $X_R \setminus \bigcup \mathscr{P}$ is open in Y. Hence $f^{\leftarrow}(X_R \setminus \bigcup \mathscr{P})$ is a nonempty open set of X_Q. As $R \neq Q$, $|f^{\to}(f^{\leftarrow}(X_R \setminus \bigcup \mathscr{P}))| = 1$, because of (1). As X_Q is anti-Hausdorff, it follows that $f^{\leftarrow}(X_R \setminus \bigcup \mathscr{P})$ is dense in X_Q. Since X is T_1 it follows that f is constant.

This completes the preparations.

Now consider any self-map f of X such that there are $a \neq b \in X$ such that $f(a) = b$. We must prove $f(x) = b$ for all $x \in X$. From (c) we see that it suffices to prove that $f(x) = b$ for $x \in X \setminus X_{(a,b)}$. So consider any $x \in X \setminus X_{(a,b)}$. As $A_{(x,a)} \cap A_{(a,b)} = \emptyset$ and $X_P = A_P \cup P$ for $P \in [\omega]^2$, we have $b \in X_{(x,a)}$. Hence $f|X_{(x,a)}$ is constant by the Claim. Therefore $f(x) = f(a) = b$, as required.

References

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